

## THE MODAL IDENTIFICATION METHOD COUPLED WITH THE ADJOINT METHOD FOR REDUCED MODEL IDENTIFICATION

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**Abstract** – We consider the resolution of an inverse problem for the identification of some reduced nonlinear diffusive models. The modal identification method is dealt with along with a gradient-type optimization method. Two distinct approaches are considered and compared. On one hand the classical finite differences approach and, on the other hand, the adjoint method for the computation of the objective function gradient. It is shown a drastic CPU time reduction when using the adjoint method, still with a high identification accuracy.

### 1. INTRODUCTION

Multidimensional nonlinear inverse heat conduction problems (IHCP) usually involve the use of a high order detailed model related to the space-time discretization. Due to the large number of degrees of freedom of the systems, the optimization algorithms used to solve the IHCP may be tremendously time-consuming. This drawback leads to identify some models which have a much lower number of degrees of freedom. Starting from the Detailed Model (DM) of the system, the modal identification method (MIM) is applied to build the corresponding Reduced Model (RM). The use of such RM allows important reduction of computation time, when solving both direct and inverse problems. However, the question of the computing time needed to obtain the RM arises.

So far, the identification of reduced models through the modal identification method was based on gradient algorithms coupled with the well-known forward finite differences method. The objective function to be minimized is the mean-squared discrepancy between the outputs of the reduced model and of the detailed model. When the direct detailed model is nonlinear, the number of iterations needed to identify the reduced models may be large leading to a high time-consumption. Moreover, the finite difference method only gives approximations of the functional gradient thus yielding to more computation time. In order to save up some time computation when building the RM, we have coupled the optimization algorithm with the adjoint method. The corresponding lagrangian penalizes the reduced space and time dependent state representation as well as the output system. The adjoint problem involves two more linear systems. The gradient of the functional resides in the end in a single matrix-vector product.

The numerical tests presented in the paper prove the ability of the adjoint methods to identify reduced models and show the drastic reduction of time computation needed to identify the reduced models.

### 2. THE DIFFUSION MODEL

Let us consider that the continuous physical phenomenon of concern is *a priori* governed by the parabolic unsteady diffusive evolution (1) in  $(x,t) \in \Omega \times I$  where  $x \in \Omega \subset \mathfrak{R}^3$  and  $t \in I = (0, t_f)$ :

Find  $\varphi = \varphi(x,t)$  such that :

$$\begin{aligned}
 \alpha \dot{\varphi} - \nabla \cdot (\beta \nabla \varphi) &= f && \text{for } (x,t) \in \Omega \times I, \\
 \varphi &= g && \text{for } x \in \partial\Omega_1, t \in I, \\
 \nabla \varphi \cdot n &= h && \text{for } x \in \partial\Omega_2, t \in I, \\
 \varphi &= \varphi_0 && \text{for } x \in \Omega, t = 0,
 \end{aligned} \tag{1}$$

where  $\alpha$  is the inertial coefficient,  $\beta$  is the diffusion coefficient,  $\nabla$  is the vector differential operator,  $n$  is the outward unit normal vector,  $\dot{\varphi} = \partial\varphi/\partial t$  where  $t$  is the marching variable, and where  $\partial\Omega_1 \cup \partial\Omega_2$  is a partition of the boundary of  $\Omega$ . Mixed Dirichlet-Neumann conditions are applied on the boundary.

After discretization of the continuous partial differential equation along with initial and boundary conditions, separating apart linear and nonlinear terms, and exhibiting the explicitly-expressed thermal inputs, the

continuous problem eqn.(1) is transformed [4,5] to the discrete one eqn.(2). In eqn.(2),  $\Phi$  is the discrete state of order  $N$ ,  $Y$  is the selected output vector of order  $q \leq N$ ,  $U$  is the explicitly-expressed input vector,  $B$  is the related input matrix,  $C$  is the matrix used to select the outputs and  $A$  is a square matrix of order  $N$ . Eventually,  $\Psi$  is the vector that exhibits all the nonlinear contributions [4].

Find  $\Phi = \Phi(t) \in \mathfrak{R}^N, Y = Y(t) \in \mathfrak{R}^q$  such that :

$$\begin{aligned} \dot{\Phi} &= A\Phi + BU + \Psi & \text{for } & t \in I, \\ \Phi &= \Phi_0 & \text{for } & t = 0, \\ Y &= C\Phi & \text{for } & t \in I, \end{aligned} \quad (2)$$

Let  $F$  be the diagonal matrix of  $A$  obtained with the use of the eigenvectors matrix  $M$ , i.e.  $F = M^{-1}AM$ . This change of variables  $\Phi = MX$  yields to solve eqn.(3) instead of eqn.(2):

Find  $X = X(t) \in \mathfrak{R}^N, Y = Y(t) \in \mathfrak{R}^q$  such that :

$$\begin{aligned} \dot{X} &= FX + GU + \Omega Z & \text{For } & t \in I, \\ X &= X_0 & \text{For } & t = 0, \\ Y &= HX & \text{For } & t \in I, \end{aligned} \quad (3)$$

where  $Z$  is the vector that contains the nonlinear combinations of states  $X_i$ . In the specific case presented downwards, the dimension of  $Z$  is  $N(N+1)/2$ , hence the dimension of the matrix  $\Omega$  is  $(N, N(N+1)/2)$ . This detailed model, expressed on its modal base is also of order  $N$ . When defining a reduced model of order  $n \leq N$  with the same structure as in eqn.(3), the direct problem reduces to:

Find  $X = X(t) \in \mathfrak{R}^n, Y = Y(t) \in \mathfrak{R}^q$  such that :

$$\begin{aligned} \dot{X} &= FX + GU + \Omega Z & \text{For } & t \in I, \\ X &= X_0 & \text{For } & t = 0, \\ \hat{Y} &= HX & \text{For } & t \in I, \end{aligned} \quad (4)$$

where the vector that contains the nonlinear combinations of states writes from now on:

$$Z = {}^t(X_1^2, X_1X_2, X_1X_3, \dots, X_1X_n, X_2^2, X_2X_3, \dots, X_2X_n, \dots, X_{n-1}^2, X_{n-1}X_n, X_n^2). \quad (5)$$

### 3. IDENTIFICATION OF THE REDUCED MODEL

All the components of the matrices involved in eqn.(4) are to be identified. The identification is performed through the resolution of an inverse problem. Normally, the searched vector parameter is defined as  $u = {}^t(F_{i,i}, G_{i,j}, \Omega_{i,k}, H_{l,i}) \in \mathfrak{R}^{n(1+p+n(n+1)/2+q)}$  with  $i = 1, \dots, n$ ,  $j = 1, \dots, p$ ,  $k = 1, \dots, n(n+1)/2$  and  $l = 1, \dots, q$ . However, taking into account that the matrix  $\hat{Y}$  is linear with respect to the matrix  $H$  when  $X$  is given, then  $H$  can be obtained using linear least squares at each time  $F$ ,  $G$  and  $\Omega$  are updated. Hence, the searched vector reduces to  $u = {}^t(F_{i,i}, G_{i,j}, \Omega_{i,k}) \in \mathfrak{R}^{n(1+p+n(n+1)/2)}$ .

The objective function to be minimized is classically the time-integrated mean-squared discrepancy between the response  $Y$  (given by the detailed model), and the response  $\hat{Y}$  given from the reduced model. Since this latter response depends on time and also on the searched parameters, it is denoted  $\hat{Y}(t; u)$  [6]. The optimization problem to be solved writes:

$$\text{Find } \bar{u} \text{ such that } j(\bar{u}) = \min_{\substack{R^1(X,u)=0, \\ R^2(\hat{Y},X,u)=0}} j(u), \quad (6)$$

with the objective function defined by:

$$j(u) = \frac{1}{2} \int_I \sum_{k=1}^q (\hat{Y}_k(t; u) - Y_k(t))^2 dt. \quad (7)$$

and where  $R^1$  and  $R^2$  represent the equations of evolution of the state variables and outputs related to the Reduced Model:

$$\begin{aligned} R^1(X, u) &= \dot{X} - FX - GU - \Omega Z = 0, \\ R^2(\hat{Y}, X, u) &= \hat{Y} - HX = 0. \end{aligned} \quad (8)$$

The adjoint method has been chosen to solve this inverse problem. The main reasons of this choice are: the adjoint (lagrangian) method is very interesting when there is no explicit dependency between the state variables involved in the objective function and the parameters; moreover, the adjoint method is interesting when the number of parameters is high. Hence, let us define the lagrangian of the problem as:

$$L(X, \hat{Y}, u, \lambda, \nu) = j(u) + (R^1(X, u), \lambda)_{A \times I} + (R^2(\hat{Y}, X, u), \nu)_{B \times I}, \quad (9)$$

where  $\lambda$  and  $\nu$  are the adjoint variables, and where the scalar products are defined as:

$$(a, b)_{A \times I} = \int_I \sum_{i=1}^n a_i b_i dt, \quad (a, b)_{B \times I} = \int_I \sum_{i=1}^q a_i b_i dt. \quad (10)$$

It may be showed that the adjoint problem related to the lagrangian eqn.(9) and to the optimization eqn.(8) is:

$$\begin{aligned} \text{Find } \lambda = \lambda(t) \in \mathfrak{R}^n, \nu = \nu(t) \in \mathfrak{R}^q \text{ such that :} \\ -\dot{\lambda} - {}^t F \lambda - {}^t \tilde{Z} {}^t \Omega \lambda - {}^t H \nu = 0 & \quad \text{for } t \in I, \\ \lambda - {}^t H \nu = 0 & \quad \text{for } t = t_f, \\ \nu = -\partial j / \partial \hat{Y} & \quad \text{for } t \in I, \\ \nu = -\partial j / \partial \hat{Y} & \quad \text{for } t = t_f. \end{aligned} \quad (11)$$

This set of adjoint equations couples one full time-dependent problem (problem in  $\lambda$ ) with one stationary-like time dependent problem (problem in  $\nu$ ). Both problems being weakly coupled (see the definitions of the different coupling types in [1]), it is possible to solve both problems in one go as:

$$\begin{aligned} \text{Find } \lambda = \lambda(t) \in \mathfrak{R}^n \text{ such that :} \\ -\dot{\lambda} - ({}^t F + {}^t \tilde{Z} {}^t \Omega) \lambda + {}^t H \partial j / \partial \hat{Y} = 0 & \quad \text{for } t \in I, \\ \lambda + {}^t H \partial j / \partial \hat{Y} = 0 & \quad \text{for } t = t_f. \end{aligned} \quad (12)$$

This adjoint problem must be solved backwards in  $t$  to be well-posed [2]. With the new variable defined by  $\tau = t_f - t$ , the adjoint problem eqn.(12) is solved forwardly. The adjoint problem eqn.(12) being solved, the objective function gradient is equal to the lagrangian gradient. With the vector of parameters defined upwards, the objective function gradient writes:

$$\nabla j(u) = {}^t \left( (X_i, \lambda_i)_{i=1, \dots, n}, (U_j, \lambda_i)_{i=1, \dots, n; j=1, \dots, p}, (Z_j, \lambda_i)_{i=1, \dots, n; j=1, \dots, n(n+1)/2} \right). \quad (13)$$

The global optimization algorithm works increasing the order  $n$  of the reduced model eqn.(4). It first starts with order  $n = 1$  to identify the components  $u = {}^t (F_{1,1}, G_{1,j}, \Omega_{1,1}) \in \mathfrak{R}^{(2+p)}$  with  $j = 1, \dots, p$ . When these components are evaluated, the algorithm identifies the components of the reduced model of order  $n = 2$ :  $u = {}^t (F_{i,i}, G_{i,j}, \Omega_{i,k}) \in \mathfrak{R}^{2(1+p+3)}$  with  $i = 1, \dots, 2$ ,  $j = 1, \dots, p$  and  $k = 1, \dots, 3$ . The order is then increased

until the below-defined global criterion (stopping rule #2) is satisfied. For each order  $n$  related to the reduced model, the optimization algorithm proceeds this way. Given an initial set of controls  $u^0$ , one builds a series defined by  $u^p = u^{p-1} + \alpha^p d^p$  where  $d^p$  is the direction of descent and  $\alpha^p$  is the descent step size. The direction of descent is given by the Broyden-Fletcher-Goldfarb-Shanno method [7]. It requires only the gradient of the objective function. Next, the optimal step size which requires both the objective function value and its gradient is given through a cubic polynomial interpolation [3]. The global procedure for the identification of the reduced model is given here-below:

Let order  $n = 1$

- (a) Let  $p = 0$ ,  $u^0$  be the starting point. Choose any positive definite matrix  $H^0$  (identity for instance)
- (b) At step  $p$ , compute the displacement direction  $d^p = -H^p \nabla j(u^p)$ , and find  $u^{p+1}$  at the minimum of  $j(u^p + \alpha d^p)$  with  $\alpha \geq 0$ .
- (c) Set  $\delta^p = u^{p+1} - u^p$  and compute  $\gamma^p = \nabla j(u^{p+1}) - \nabla j(u^p)$  to actualize:
$$H^{p+1} = H^p + \frac{\delta^{p \prime} \delta^p}{\delta^{p \prime} \gamma^p} - \frac{H^p \gamma^{p \prime} \gamma^p H^p}{\gamma^{p \prime} H^p \gamma^p}$$
- (d) Stopping rules #2 (see below). If not satisfied, set  $p \leftarrow p + 1$  and return to (b)

Stopping rule #1 (see below). If satisfied: end. Else set  $n \leftarrow n + 1$  and return to (a)

The stopping rule #1 is used to stop the incrementation of the order  $n$  of the reduced model, that is when a global stabilization of the objective function is reached. The stopping rule #2 is used to stop the identification algorithm for a given order  $n$ . These stopping criteria are listed in [4,5].

Let us point out that the quality related to the RM is bounded to the quality of the DM used for data computation. Thus, a coarse DM yields to a coarse RM, and an accurate DM yields to an accurate RM. Moreover, in contrast to linear systems for which a RM identified from responses to any known input signal will be a priori valid for any other input signal, nonlinear systems basically react in a different way according to the excitation level. Actually, a RM identified from data given from a given input signal  $U_1$  will not necessary adequately reproduce the system's behaviour when a different input signal  $U_2$  is applied. The signal used to generate data for the RM identification must allow the system to react in large ranges of states levels and frequencies.

#### 4. THE NUMERICAL RESULTS

This section presents the numerical results and especially the validation of the proposed approach. The test case deals with a 3 dimensional case with the nonlinearity given by the diffusion coefficient dependent on the state as  $\beta(X) = 16(1 + 0.01(X - 20))$ . The inertial coefficient  $\alpha$  equals  $= 4.029 \cdot 10^6$  SI. Boundary conditions are given by eqn.(14), where  $h$  equals 50 SI.

$$\begin{aligned}
 -\beta_{x_1} X &= U & \text{for } x_1 = 0, t \in I, \\
 -\beta_{x_1} X &= hX & \text{for } x_1 = 0.1, t \in I, \\
 X &= 0 & \text{for } x_2 = 0, t \in I, \\
 \beta_{x_2} X &= 0 & \text{for } x_2 = 0.1, t \in I, \\
 -\beta_{x_3} T &= hX & \text{for } x_3 = 0, t \in I, \\
 \beta_{x_3} X &= 0 & \text{for } x_3 = 0.1, t \in I,
 \end{aligned} \tag{14}$$

The initial condition is given by the resolution of the steady state regime when boundary conditions given by eqn.(14) are applied with a given constant loading  $U$ . The domain is discretized using the Finite Volumes Method [8], with 11 nodes in each direction, leading to a DM given by eqn.(2) of order  $N = 1331$ .

To illustrate the method, we consider three points respectively located at  $(1;5;2) \cdot 10^{-2}$ ,  $(5;5;5) \cdot 10^{-2}$  and  $(9;5;8) \cdot 10^{-2}$ .

The signal  $U_1$  used for the reduced model identification is presented in Figure 1. The number of steps  $nt$  equals 10800. Each step lasts 5 seconds of time. In total, one thus gets 32403 points of state evolution for the three considered spatial points. Figure 2 shows the response obtained with the DM of order  $N = 1331$ .

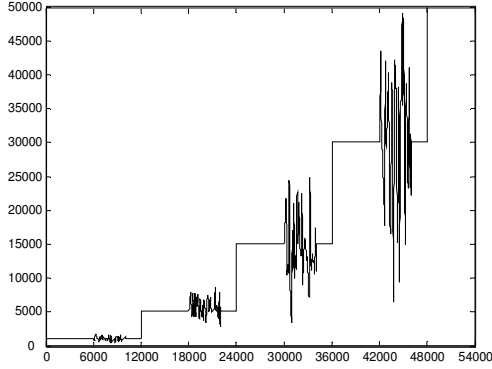


Figure 1: Flux density  $U_1(t)$  used for RM identification (both  $U_1$  and  $t$  are in SI).

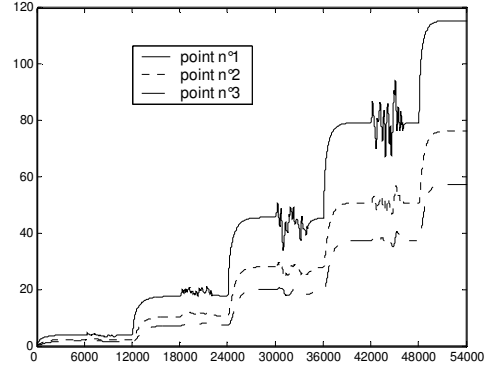


Figure 2: DM's state responses (in SI) at the three points when the input signal shown in Figure 1 for the RM identification is applied. These states along with the corresponding signal are used as data for the RM identification.

The numerical results of the identification of RMs of orders  $n = 1$  to 5 are summarized in Table 1. For each distinct order  $n$ , the minimization of the quadratic criterion is performed. The identification has been performed using two different approaches to compute the gradient of  $j(u)$ : the classical Finite Differences Method and the Adjoint Method as developed in this paper. At first, some similar observations can be made for both methods. For the three first orders from  $n=1$  to  $n=3$ , the mean quadratic error  $\sigma_Y^{id}$  (see [4]) characterizing the RM identification quality rapidly decreases from 1.4 SI to 0.02 SI. For  $n=4$ , the gain in precision is still substantial with  $\sigma_Y^{id} = 0.01$  SI. Increasing the order to  $n=5$  leaves  $\sigma_Y^{id}$  quasi-unchanged: the identification criterion is slightly better but the improvement is not significant.

Table 1: RM identification results. Comparison between Finite Differences and Adjoint Method for the computation of  $\nabla j(u)$ .

| Reduced Model order $n$ | $\nabla j(u)$ computed by Finite Differences |                      |                        |                                    | $\nabla j(u)$ computed by Adjoint Method |                      |                        |                                    |
|-------------------------|--|----------------------|------------------------|------------------------------------|--|----------------------|------------------------|------------------------------------|
|                         | $\sigma_Y^{id}$                              | Number of iterations | Computing Time (s CPU) | Average time per iteration (s CPU) | $\sigma_Y^{id}$                          | Number of iterations | Computing Time (s CPU) | Average time per iteration (s CPU) |
| 1                       | 1.396  | 12                   | 7                      | 0.6                                | 1.396                                    | 10                   | 34                     | 3.4                                |
| 2                       | 0.187  | 73                   | 134                    | 1.8                                | 0.187                                    | 76                   | 368                    | 4.8                                |
| 3                       | $1.92 \cdot 10^{-2}$                         | 343                  | 1646                   | 4.8                                | $1.96 \cdot 10^{-2}$                     | 207                  | 967                    | 4.7                                |
| 4                       | $9.29 \cdot 10^{-3}$                         | 277                  | 9263                   | 33.4                               | $1.02 \cdot 10^{-2}$                     | 111                  | 849                    | 7.6                                |
| 5                       | $8.98 \cdot 10^{-3}$                         | 4                    | 364                    | 91                                 | $1.01 \cdot 10^{-2}$                     | 1                    | 17                     | 17                                 |

Table 1 also allows a comparison of computing time for the RM identification when using the Finite Difference Method on one hand, and the Adjoint Method on the other hand for the computation of the objective function gradient. In terms of average time per iteration, it can be observed that for orders  $n=1$  and  $n=2$ , the Finite Differences Method is faster. For  $n=3$ , both methods are almost as efficient, and for  $n=4$ , the Adjoint Method becomes much faster. Even though the Adjoint Method is, in this specific case, more efficient than the Finite Difference Method only for orders greater than 2, cumulating for each method the total times needed to identify the reduced model gives the advantage to the Adjoint Method. For instance the total CPU times to access the third order RMs are equal to 1369 and 1787 second CPU for respectively the Adjoint Method and the Finite Difference Method. To access the fourth order RM, the total CPU times are respectively equal to 2218 and 11050 second CPU.

In order to validate the identified RM, it is necessary to test it with an input signal  $U_2$  very different from the signal used for the model identification. Figure 3 shows the signal  $U_2$  including steps, ramps and a sinusoid.

Figure 4 shows the state responses computed with the DM. For both approaches, the RM reproduces very efficiently the DM's behaviour. Since it is not possible to distinguish the quasi-perfectly superposed curves, we rather propose to show the discrepancies between both models on separated graphs. The discrepancies are shown in both Figure 5 and Figure 6.

Figure 5 shows the residuals between DM's responses and those computed with the order 4 RM obtained using the Finite Differences Method. Figure 6 shows the residuals between DM's responses and those computed with the order 4 RM obtained using the Adjoint Method. Discrepancies are of the same order of magnitude for both approaches. The RM built using the Finite Difference Method seems slightly better for the point n°1 while the RM built using the Adjoint Method is better for the point n°2. Though the quality of the RM is almost identical for both approaches, the main difference between both approaches comes from the high time reduction when using the Adjoint Method.

Next, it is pointed out that the use of the RM instead of the DM yields to a drastic reduction of computation time. The direct problem resolution requires only 0.15 s CPU with a RM of order 4, instead of 163 s CPU with the original DM of order 1331. This reduction factor is greater than 1000.

Eventually, one should note the quite high level of nonlinearities in the proposed example: if the linear RM obtained by zeroing the nonlinear term  $\Omega Z(X(t))$  involved in (4) is used, resulting state evolutions are far from those shown on Figure 4, with discrepancies up to 80 SI.

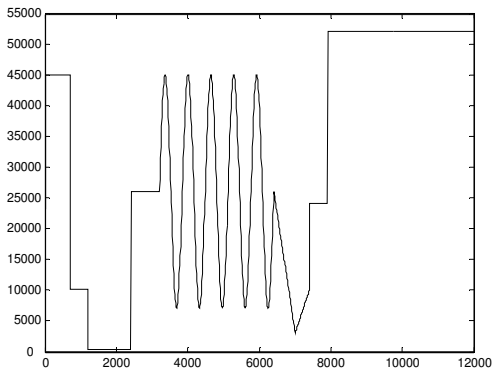


Figure 3: Test heat flux density  $U_2$  used for RM validation.

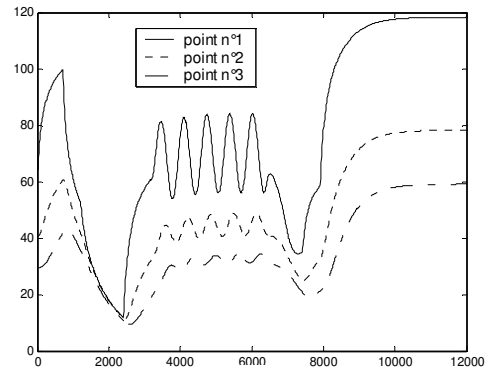


Figure 4: DM's temperature responses at points N°1, 2 and 3 when the test input  $U_2$  is applied.

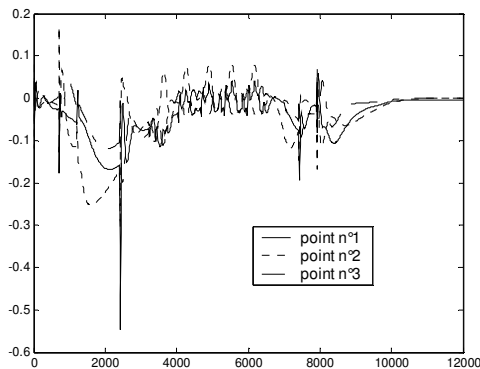


Figure 5 : Discrepancies between DM's and RM's (FDM) responses when test function is applied.

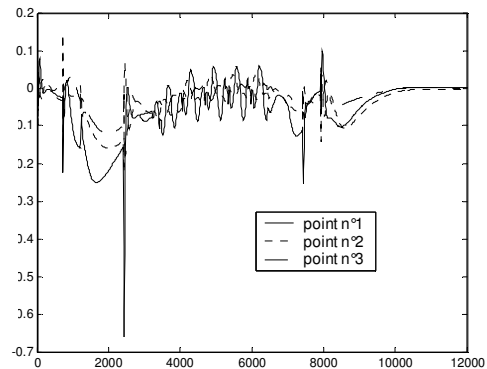


Figure 6 : Discrepancies between DM's and RM's (AM) responses when test function is applied.

## 5. CONCLUSIONS

The combination of the Modal Identification Method for identifying some Reduced Models and the Adjoint Method for computing gradients has been developed in this paper. The adjoint problem has been derived from the structure of the direct model.

Both approaches (finite differences method and the adjoint method) have led to similar results in terms of Model Reduction quality. However, the high reduction of CPU when identifying the RM with the Adjoint Method comes from at least two reasons. At first, the adjoint problem is linear even though the direct model is

not. Next, because the adjoint problem must be solved only once (whatever the number of parameters to be identified).

The accuracy of identified Reduced Models has also been shown, with drastic reduction of computation time when using the Reduced Models instead of the original Detailed Model.

Further developments include the extension of the proposed approach to the case where both the inertial and the diffusive coefficients are state dependent. Next studies shall concern the model reduction for fluid mechanics problems where a large number of parameters are to be identified.

## REFERENCES

1. Y. Favennec, V. Labbé and F. Bay, The ultraweak time coupling in nonlinear multiphysics modelling and related optimization problems. *Int. J. Numer. Meth. Engng.* (2004) **60**, 803-831.
2. Y. Favennec, V. Labbé and F. Bay, Induction heating processes optimization – a general optimal control approach. *J. Comput. Phys.* (2003) **187**, 68-94.
3. P.E. Gill, W. Murray and M.H. Wright, *Practical Optimization*, Academic Press, London, 1992.
4. M. Girault and D. Petit, Identification methods in nonlinear heat conduction – Part 1: model reduction. *Int. J. Heat Mass Transfer* (2005) **48**, 105-118.
5. M. Girault and D. Petit, Identification methods in nonlinear heat conduction – Part 2: inverse problem using a reduced problem. *Int. J. Heat Mass Transfer* (2005) **48**, 119-133.
6. Y. Jarny, M.N. Ozisik and J.P. Bardou, A general optimization method using adjoint equation for solving multidimensional inverse heat conduction. *Int. J. Heat Mass Transfer* (1991) **34**, 2911-2919.
7. M. Minoux, *Mathematical Programming – Theory and Algorithms*, Wiley and Sons, Chichester, 1986.
8. S.V. Patankar, *Numerical Heat Transfer and Fluid Flow*, Series in computational methods in mechanics and thermal sciences, Taylor & Francis, Hemisphere Pub, 1980.